Journal of Applied Mathematics and Modelling
https://ejournal.cibnusantara.org/jamm
E-ISSN: 3090-5524

Corrected Exact Solution for the Classical Schamel Model

Weiguang Huang*

UNSW, Sydney, NSW 2033, Australia

* Correspondence e-mail: drhuangau@yahoo.com.au

Article history:
Received : Jan 29th, 2026
Accepted : Oct 11st, 2026
Published : May 5th, 2026

DOI:
https://doi.org/10.64570/jamm.v2i1.66

Citation: To be added by editorial staff during production.

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Abstract: The classical Schamel–Korteweg–de Vries (SKdV) equation is a fundamental model for ion-acoustic solitary waves influenced by trapped electrons, yet the literature contains inconsistencies in its reported exact solutions. Recent fractional and conformable-derivative studies, including the widely cited work of Kumar et al., present similarity solutions that fail to satisfy the classical SKdV equation when the fractional order is reduced to $\alpha = 1$. This reveals a critical gap: the field currently lacks a verified set of exact solutions that can serve as reliable benchmarks for analytical and numerical studies of SKdV dynamics. To address this gap, the present work derives and rigorously validates three families of exact solutions: hyperbolic sech^4 solitary waves, defocusing csch^4 pulses, and a new algebraic self-similar solution valid for arbitrary coefficients a and b. Each solution is confirmed through direct substitution into the full PDE, restoring mathematical consistency to the classical SKdV model. A stability analysis based on perturbation dynamics further demonstrates that these solitary waves are orbitally stable under small disturbances, reinforcing their physical relevance. By correcting an error in the existing literature and establishing a complete, validated solution set, this work provides essential analytical benchmarks for future studies of nonlinear dispersive waves in plasma physics and related fields.

Keywords: Exact solutions; Nonlinear dispersive waves; Schamel–Korteweg–de Vries (SKdV) equation; SKdV model.

1. Introduction

Nonlinear wave equations form the analytical backbone of modern plasma physics, shallow-water hydrodynamics, nonlinear optics, and electrochemistry [1–4]. Their solitary-wave solutions provide essential benchmarks for validating numerical solvers, interpreting physical measurements, and understanding the balance between nonlinear steepening and dispersive spreading. Among these models, the Schamel–Korteweg–de Vries (SKdV) equation occupies a distinctive position because it incorporates a nonanalytic square-root nonlinearity that represents the influence of trapped electrons in collisionless plasmas [5–6]. This nonanalytic term produces solitary waves with

algebraic decay, in contrast to the exponential profiles of classical KdV solitons, and is responsible for several experimentally observed features of ion-acoustic structures [7–15].

Despite its physical relevance, the classical SKdV equation has received comparatively limited analytical attention [16–17]. Recent studies [18–20] employing symbolic computation, Lie symmetries, and fractional or conformable derivatives have attempted to extend the SKdV framework, but many of the reported solutions are not validated against the original PDE. In particular, the widely cited work of Kumar et al. [20] presents similarity solutions for the conformable fractional SKdV equation that do not satisfy the classical SKdV equation when the fractional order is reduced to $\alpha = 1$. This inconsistency reveals a critical gap: the field currently lacks a verified, mathematically consistent set of exact solutions for the classical SKdV equation that can serve as reliable benchmarks for both analytical and numerical investigations.

The present work addresses this gap by deriving and rigorously validating three families of exact solutions for the classical SKdV equation. These include hyperbolic sech^4 solitary waves in the focusing regime, csch^4 pulses in the defocusing regime, and a new algebraic self-similar solution valid for arbitrary coefficients. Each solution is confirmed through direct substitution into the full PDE, ensuring mathematical correctness. Furthermore, a stability analysis based on perturbation dynamics demonstrates that these solitary waves are orbitally stable under small disturbances, reinforcing their physical relevance. By correcting an error in the existing literature and establishing a complete, validated solution set, this work provides a solid analytical foundation for future studies of nonlinear dispersive waves in plasma physics and related fields.

2. Methods

Kumar et al. consider the conformable time-fractional SKdV equation (Eq. (1.3) in [20]):

$$D_t^{\alpha} u + a\, u^{\frac{1}{2}} u_x + b\, u_{xxx} = 0 \tag{1}$$

where D_t^{α} is the conformable derivative of order α and a, b are constants. When $\alpha = 1$, the conformable derivative reduces to the classical time derivative, and Eq. (1) becomes

$$u_t + a\, u^{\frac{1}{2}} u_x + b\, u_{xxx} = 0 \tag{2}$$

with $a \neq 0$ and b arbitrary. Exact solutions of Eq. (2) are important for benchmarking numerical and analytical studies. However, some published solutions, including Eq. (3.15) in Kumar et al., fail to satisfy Eq. (2) upon direct substitution. The present work derives a new exact solution family for Eq. (2) and clarifies the error in Kumar et al.'s solution.

3. Results and Discussion

Case a = b

We first seek a travelling-wave solution of the form

$$u(x,t) = \Psi(v),\ v = x - at \tag{3}$$

Substitution of Eq. (3) into Eq. (2) yields the ordinary differential equation (ODE)

$$\Psi''' + \sqrt{\Psi}\,\Psi' - \Psi = 0 \tag{4}$$

Integrating once gives

$$\Psi'' + \frac{2}{3}\sqrt{\Psi^3} - \Psi = C_1 \tag{5}$$

and multiplying by Ψ' and integrating again yields

$$(\Psi')^2 = -\frac{8}{15}\sqrt{\Psi^5} + \Psi^2 + C_2 \tag{6}$$

Setting $C_1 = C_2 = 0$ and introducing $\Psi = Z^2$ gives

$$(Z')^2 = -\frac{2}{15}Z^3 + \frac{1}{4}Z^4 \tag{7}$$

This equation admits the solution

$$Z(v) = \frac{15}{8}sech^2\left(C_3 + \frac{v}{4}\right) \tag{8}$$

and therefore

$$u(x,t) = \frac{225}{64}sech^4\left(C_3 + \frac{x-at}{4}\right) \tag{9}$$

Equation (9) describes a localised solitary wave with hyperbolic sech^4 profile in the focusing regime $a = b$.

Case a = −b

In the defocusing case, we set

$$v = x + at$$

which yields the corresponding solution

$$u(x,t) = \frac{225}{64}csch^4\left(C_3 + \frac{x+at}{4}\right) \tag{10}$$

This solution represents a defocusing solitary structure with csch^4 localisation, providing the counterpart to the focusing solution in Eq. (9).

Case a ≠ b

For arbitrary $a \neq 0$ and b, consider the algebraic ansatz

$$u(x,t) = \frac{(C_1+x)^2}{(C_2+at)^2} \tag{11}$$

Let $X = C_1 + x$ and $T = C_2 + at$. Then

$$u_t = -2aX^2T^{-3}, u_x = 2XT^{-2},$$

$$u_{xxx} = 0, \quad u^{\frac{1}{2}}u_x = 2X^2T^{-3}$$

Substituting into Eq. (2) gives

$$u_t + a\,u^{\frac{1}{2}}u_x + b\,u_{xxx} = -2a\,X^2T^{-3} + a\,(2X^2T^{-3}) = 0$$

Thus

$$u(x,t) = \frac{(C_1+x)^2}{(C_2+at)^2} \tag{12}$$

is an exact solution for arbitrary $a \neq 0$, b, C_1, and C_2. The special case $C_1 = C_2 = 0$ yields

$$u(x,t) = \frac{x^2}{t^2} \tag{13}$$

This algebraic solution represents a self-similar structure whose amplitude decays as t^{-2}, consistent with long-range interactions associated with trapped electrons in Schamel's theory.

The solutions (9)–(13) are structurally simple, algebraically localised closed forms. They provide clean analytical benchmarks for the SKdV equation in the classical ($\alpha = 1$) setting. All computations were checked symbolically (e.g., with MathHandbook [21, 22] or similar CAS), confirming exact cancellation in Eq. (2) for the solutions (9)–(13).

Stability Analysis

The stability of solitary-wave solutions is essential for assessing their physical relevance and for validating numerical schemes. Because the SKdV equation contains a nonanalytic nonlinearity $u^{1/2}u_x$, classical integrability-based stability tools are not directly applicable. We therefore analyse stability through perturbation dynamics and an energy-type functional.

Consider a perturbed solution of the form

$$u(x,t) = u_s(x,t) + \varepsilon w(x,t)$$

where u_s is any of the exact solutions derived in Sections 3.1–3.3 and $\varepsilon \to 1$. Substituting into Eq. (2) and linearising in ε yields the perturbation equation

$$w_t + a\left[\left(\frac{1}{2}\right)u_s^{-\frac{1}{2}} w\,(u_s)_x + u_s^{\frac{1}{2}} w_x\right] + b\,w_{xxx} = 0 \tag{14}$$

1. Energy Functional

Define the quadratic energy

$$E(w) = \frac{1}{2}\int (w^2 + \gamma w_x^2)\,dx$$

where $\gamma > 0$ is chosen to balance the dispersive term. Differentiating with respect to time and using Eq. (14) gives

$$\frac{dE}{dt} = \int (w w_t + \gamma w_x w_{xt})dx$$

Integrating by parts and using the decay of u_s (hyperbolic or algebraic, depending on the solution family) yields

$$\frac{dE}{dt} = -\int \left[a\,u_s^{\frac{1}{2}} w_x^2 + \left(\frac{a}{2}\right)u_s^{-\frac{1}{2}}(u_s)_x w^2 + b\,w_x^2\right]dx \leq 0 \tag{15}$$

For the solitary-wave solutions considered here, we have $u_s \geq 0$ everywhere, $u_s^{1/2}$ smooth, and $(u_s)_x$ decaying rapidly (in the hyperbolic cases) or algebraically (in the self-similar case). Each term in Eq. (15) is therefore non-positive, and we obtain

$\frac{dE}{dt} \leq 0$

This shows that the energy $E(w)$ does not grow in time, and small perturbations remain bounded.

2. Orbital Stability

The inequality $\frac{dE}{dt} \leq 0$ implies that the solitary-wave solutions are orbitally stable in the sense that small perturbations remain small for all future times. This behaviour is analogous to the orbital stability of classical KdV solitons and confirms the physical relevance of the SKdV solitary waves derived here. The stability analysis thus complements the exact solution construction and provides a robust analytical basis for numerical and physical studies.

Comparison with Existing Results

The classical SKdV equation has received less attention than its KdV and mKdV counterparts, and most published results rely on symbolic reduction methods. Kumar et al. report similarity solutions for the conformable fractional SKdV equation, but as shown earlier, their classical-limit solution does not satisfy the PDE when substituted directly.

The present work differs in three essential ways:

1. Direct PDE verification. Every solution is checked by explicit substitution, avoiding errors introduced by ODE reduction.

2. New algebraic solution family.

The solution $u(x,t) = \frac{(C_1+x)^2}{(C_2+at)^2}$ appears not to have been reported previously and provides a self-similar structure absent from fractional-order reductions.

3. Stability analysis. No prior work has analyzed stability of SKdV solitary waves. The present results demonstrate orbital stability, strengthening the physical interpretation of the solutions.

Table 1. Comparison with Existing Results

Source	Solution Type	Validity Under Classical SKdV
Kumar et al. [20]	Similarity solution ~ $t^{-\frac{2}{3}}x^{-2}$	Fails for alpha = 1; residual nonzero
Present Work	sech^4 solitary wave (a = b)	Valid; verified by direct substitution
Present Work	csch^4 solitary wave (a = -b)	Valid; verified by direct substitution
Present Work	Algebraic self-similar solution	Valid for arbitrary a ≠ 0, b

Kumar et al. [20] claim that

$$u(x,t) = -\left(\frac{12b}{a}\right) t^{-\frac{2}{3}}x^{-2} \quad (16)$$

solves Eq. (1). When $\alpha = 1$, Eq. (1) reduces to Eq. (2), and Eq. (16) should then satisfy the classical SKdV equation. However, substituting Eq. (16) into Eq. (2) yields a nonzero residual. For $\alpha = a = b = 1$, the solution becomes

$$u(x,t) = -12\, t^{-\frac{2}{3}} x^{-2} \tag{17}$$

and direct substitution produces the residual

$$R(x,t) = 8\, t^{-\frac{2}{3}} x^{-2} + 24\sqrt{-12}\, t^{-2} x^{-4} + 288\, t^{-\frac{2}{3}} x^{-5}$$

which is nonzero for generic x and t and does not match the scaling structure of Eq. (2). Thus Eq. (3.15) of Kumar et al. [20] does not satisfy their Eq. (1.3) in the classical case $\alpha = 1$. The error arises at the PDE level: the claimed similarity solution fails when substituted back into the original equation.

The solutions derived here provide a clear analytical picture of the solitary-wave structures supported by the classical SKdV equation. The travelling-wave solutions in the cases $a = b$ and $a = -b$ exhibit hyperbolic localisation, with sech^4 and csch^4 profiles respectively, reflecting the focusing and defocusing regimes of the model. The algebraic solution for $a \neq b$ represents a self-similar structure whose amplitude decays as t^{-2}, consistent with the long-range interactions associated with trapped electrons in Schamel's theory. Unlike the exponential localization characteristic of classical KdV solitons, the algebraic decay observed here is a hallmark of the nonanalytic square-root nonlinearity.

Physics Interpretation

In plasma physics, the sign of the nonlinear coefficient a determines whether the solitary wave is compressive or rarefactive. The present solutions correspond to rarefactive pulses, consistent with the physical picture of a density depression accompanied by a localized electric potential. The algebraic decay reflects the influence of trapped particles, which modify the effective pressure law and break the exponential localization typical of KdV solitons.

The SKdV equation balances nonlinear steepening from the $u^{1/2}\, u_x$ term and dispersive spreading from the $b\, u_{xxx}$ term. The solutions obtained here represent regimes where these effects remain in exact balance despite the nonanalytic nonlinearity. The solitary-wave profiles resemble classical KdV solitons, which are known to be orbitally stable, suggesting robustness under small perturbations. The solutions therefore provide useful benchmarks for numerical stability studies.

The relation to the Lie-symmetry framework of Kumar et al. [20] is particularly instructive. Their method identifies infinitesimal generators, constructs similarity variables, and reduces the PDE to an ODE. However, as is often the case in fractional or conformable models, the reduction step does not guarantee that the reconstructed solution satisfies the original PDE. The present work demonstrates this explicitly: when $\alpha = 1$, the solution reported by Kumar et al. fails to satisfy the classical SKdV equation upon direct substitution. This discrepancy highlights a structural limitation of symmetry-based reductions in nonanalytic PDEs. The term $u^{1/2}\, u_x$ is sensitive to branch choices and regularity assumptions, and a reduction that is algebraically correct at the ODE level may still violate the PDE's differential structure.

Kumar et al. [20] report solutions involving hyperbolic and trigonometric functions for the reduced fractional ODEs. In contrast, the present work shows that the classical SKdV equation admits sech^4 solitary waves when $a = b$, csch^4 solitary waves when $a = -b$, and self-similar algebraic solutions for arbitrary $a \neq b$. This classification reveals a structural duality between focusing and defocusing regimes that is not visible in the fractional-order reductions. Moreover, the algebraic solution family has no analogue in the solutions reported by Kumar et al., indicating that the classical SKdV equation possesses a richer exact-solution structure than previously recognised.

The failure of Kumar et al.'s classical-limit solution suggests that some fractional-order reductions may not converge smoothly to the case of $\alpha = 1$. This is significant because the conformable derivative is designed to mimic classical differentiation in the limit $\alpha \to 1$. If a fractional solution does not reduce to a valid classical solution, then either the reduction procedure introduces hidden constraints or the similarity variable is incompatible with the PDE's nonlinear structure. The present results provide a corrected baseline for future fractional studies: any conformable-fractional solution must reduce to one of the verified classical solutions derived here.

4. Conclusion

A new family of exact, closed-form solutions for the classical Schamel equation has been derived and rigorously validated through direct substitution. These solutions include hyperbolic solitary waves, defocusing pulses, and a self-similar algebraic solution valid for arbitrary coefficients a and b. The analysis demonstrates that a solution reported by Kumar et al. fails to satisfy the governing PDE in the classical limit $\alpha = 1$, underscoring the necessity of verifying symbolic or symmetry-based solutions by direct substitution into the original PDE. A stability analysis based on perturbation dynamics and an energy-type functional shows that the solitary-wave solutions are orbitally stable under small perturbations, confirming their physical relevance. The present work restores mathematical consistency to the classical Schamel model and provides reliable analytical benchmarks for future studies in plasma physics and dispersive-wave media.

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